

GENERALIZING AXIOMS OF r -PLANES AND r -SPHERES ON RIEMANNIAN AND KÄHLER MANIFOLDS

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ABSTRACT. The famous theorems of Cartan, related to the axiom of r -planes, and Leung-Nomizu about the axiom of r -spheres were extended to Kähler geometry by several authors. In this paper we replace the strong notions of totally geodesic submanifolds (r -planes) and extrinsic spheres (r -spheres) by a wider class of special isometric immersions such that theorems of type “axioms of r -special submanifolds” could hold. We verify also that there are plenty of special submanifolds in real and complex space forms and, in the codimension one case, in Einstein manifolds. In the proof of our theorem in the complex case, a new class of Kähler manifolds arose naturally, which we named XY -manifolds. They satisfy the symmetry property $\langle \bar{R}(X, JX)Y, JX \rangle = \langle \bar{R}(Y, JY)X, JY \rangle$, where J is the almost complex structure, \bar{R} is the curvature tensor and X, JX, Y, JY are orthonormal tangent vectors.

1. INTRODUCTION

An m -dimensional Riemannian manifold M is said to satisfy the axiom of r -planes if there exists $2 \leq r \leq m-1$ such that, for any $p \in M$ and any r -dimensional linear subspace W of the tangent space $T_p M$, there exists a totally geodesic submanifold S with $T_p S = W$. Cartan proved that any manifold satisfying this axiom has constant sectional curvature ([Ca]).

A umbilical submanifold of a Riemannian manifold M is called an extrinsic sphere when it has parallel mean curvature vector. One says that an m -dimensional Riemannian manifold M satisfies the axiom of r -spheres if there exists $2 \leq r \leq m-1$ such that for any $p \in M$ and any r -dimensional linear subspace W of $T_p M$ there exists an extrinsic sphere S with $T_p S = W$. Leung and Nomizu extended Cartan’s theorem proving that if M satisfies the axiom of r -spheres then it has constant sectional curvature ([LN]).

A Hermitian manifold M of real dimension $2m$ is said to satisfy the axiom of holomorphic $2r$ -spheres if there exists $1 \leq r \leq m-1$ such that for any $p \in M$ and any $2r$ -dimensional holomorphic subspace W of $T_p M$ there exists an extrinsic sphere S satisfying $T_p S = W$. Similarly, M satisfies the axiom of antiholomorphic r -spheres if there exists $2 \leq r \leq m-1$ such that for any $p \in M$ and any r -dimensional antiholomorphic subspace W of $T_p M$ there exists an extrinsic sphere S satisfying $T_p S = W$. If M is a Kähler manifold, any of these two hypotheses imply that M has constant holomorphic curvature ([CO], [G], [GM], [H], [MY]).

This paper was motivated by the following question: what could be the most general extension of the notions of totally geodesic submanifolds and extrinsic spheres

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so that results similar to the above theorems would still hold? First one could ask if some theorem involving an axiom of r -minimal submanifolds could be true. However, we expect that the answer for this question is negative. We even conjecture that, given any point p in a Riemannian manifold M , any linear subspace $W \subset T_p M$ and any vector v orthogonal to W , there exists a submanifold S tangent to W at p having parallel mean curvature vector H with $H(p) = v$. In the case $v = 0$ this could be called an “infinitesimal Dirichlet problem”.

We will define a class of isometric immersions which contains the totally geodesic submanifolds and the extrinsic spheres. Such a class satisfies a theorem of type “axiom of r -special immersions”.

We first fix some notations and definitions. Fix an isometric immersion $f : S \rightarrow M$ and $p \in S$. By $(T_p S)^\perp$ we denote the orthogonal complement of $T_p S$ relatively to $T_p M$. If $Z \in T_p M$ we denote by Z^\perp the projection of Z onto $(T_p S)^\perp$. If $\bar{\nabla}$ is the Levi-Civita connection of M , X is a vector field on S and η is a vector field on M which is orthogonal to S , we set $\nabla_X^\perp \eta = (\bar{\nabla}_X \eta)^\perp$. Let α denote the second fundamental form of f , namely, we write $\alpha(X, Y) = (\bar{\nabla}_X Y)^\perp$, where X, Y are vector fields on S . We recall that an orthonormal frame (X_i) on a neighborhood of p in S is said to be geodesic at p if $(\nabla_{X_i} X_j)(p) = 0$ for all i, j , where ∇ is the Levi-Civita connection of S .

Definition 1.1. *Let $f : S \rightarrow M$ be an isometric immersion. We say that f is special if, for any $p \in S$ and any orthonormal basis v_1, \dots, v_r of $T_p S$, it holds that*

$$(1) \quad \sum_i \bar{R}(v_i, w)v_i \in T_p S,$$

for all $w \in T_p S$, where $\bar{R}(X, Y) = \bar{\nabla}_X \bar{\nabla}_Y - \bar{\nabla}_Y \bar{\nabla}_X - \bar{\nabla}_{[X, Y]}$ is the curvature tensor of M .

Remark 1. We will see in Lemma 3.2 below that Equation (1) does not depend on the choice of the orthonormal basis v_1, \dots, v_r of $T_p S$.

Now we present a stronger definition.

Definition 1.2. *Let $f : S \rightarrow M$ be an isometric immersion where S is an r -dimensional manifold. We say that f is very special if for any $p \in S$ and any vectors $v, w \in T_p S$, it holds that*

$$(2) \quad \bar{R}(v, w)v \in T_p S.$$

Remark 2. It follows from the definition that any immersed curve is very special.

Remark 3. We will see in Proposition 3.1 below that, for dimensions greater than 1, umbilical submanifolds are very special if, and only if, they are extrinsic spheres, hence the class of very special isometric immersions contains the extrinsic spheres. Note that this is false in the case that the dimension of S is 1, since in this situation all immersions are very special and umbilical, but not necessarily extrinsic spheres.

Definition 1.3. *Given a Riemannian manifold M of dimension $m \geq 3$ and $2 \leq r \leq m - 1$, we will say that M satisfies the axiom of special (respectively, very special) r -submanifolds if, for any $p \in M$ and any r -dimensional linear subspace W of $T_p M$, there exists a special (respectively, very special) submanifold S satisfying $T_p S = W$.*

Definition 1.4. Let M be a Kähler manifold of real dimension $2m$, $m \geq 2$. We define similar axioms in complex geometry:

- (a) Given $1 \leq r \leq m - 1$, we say that M satisfies the axiom of special holomorphic $2r$ -submanifolds if, for any $p \in M$ and any $2r$ -dimensional holomorphic linear subspace W of $T_p M$, there exists a special submanifold S satisfying $T_p S = W$;
- (b) Given $2 \leq r \leq m$, we say that M satisfies the axiom of special antiholomorphic r -submanifolds if, for any $p \in M$ and any r -dimensional antiholomorphic linear subspace W of $T_p M$, there exists a special submanifold S satisfying $T_p S = W$.

Our first result is the following extension of the Theorems of Cartan and Leung-Nomizu.

Theorem 1.1. Let M be an m -dimensional Riemannian manifold with $m \geq 3$. Then it holds that:

- (a) If M satisfies the axiom of special r -submanifolds for some $2 \leq r \leq m - 1$, then M has constant sectional curvature if $r \leq m - 2$ and M is Einstein if $r = m - 1$;
- (b) If M satisfies the axiom of very special $(m - 1)$ -submanifolds, then M has constant sectional curvature.

Reciprocally, any isometric immersion in a manifold of constant sectional curvature is very special and any hypersurface in an Einstein manifold is special.

Remark 4. Note that Einstein manifolds show that Item (a) in Theorem 1.1 may not be improved to obtain constant sectional curvature in the case $r = m - 1$.

Before stating our similar result in Kähler geometry (Theorem 1.2 below) we will introduce the class of XY -manifolds, which has an important tool in the proof of this theorem.

Definition 1.5. A Kähler manifold M of real dimension $2m$, $m \geq 2$, with almost complex structure J is said to be a XY -manifold if for any local orthonormal vector fields X, JX, Y, JY it holds that

$$(3) \quad \langle \bar{R}(X, JX)Y, JX \rangle = \langle \bar{R}(Y, JY)X, JY \rangle.$$

It is well known that the assumption that $\langle \bar{R}(X, JX)Y, JX \rangle = 0$ is equivalent to the fact that M has constant holomorphic sectional curvature (see Proposition 6.1 below). Thus the class of XY -manifolds includes Kähler manifolds with constant holomorphic sectional curvature. We would like to propose the following

Conjecture 1. There exist examples of XY -manifolds with non-constant holomorphic sectional curvature.

In Kähler geometry we present the following extension of the results in [CO], [G], [GM], [H], [MY].

Theorem 1.2. Let M be a Kähler manifold of real dimension $2m$, $m \geq 2$. Then M has constant holomorphic sectional curvature if one of the following conditions hold:

- (a) M satisfies the axiom of special holomorphic $2r$ -submanifolds, for some $1 \leq r \leq m - 1$;

- (b) M satisfies the axiom of special antiholomorphic r -submanifolds, for some $2 \leq r \leq m$.

Reciprocally, all complex and all totally real immersions in a Kähler manifold of constant holomorphic sectional curvature are very special.

2. SOME EXAMPLES

Since any curve is very special, if we take immersed curves $f_i : (a_i, b_i) \rightarrow M_i$, $i = 1, \dots, n$, then $F : (a_1, b_1) \times \dots \times (a_n, b_n) \rightarrow M_1 \times \dots \times M_n$, given by $F(t_1, \dots, t_n) = (f_1(t_1), \dots, f_n(t_n))$ is a very special immersion. Now we list some other examples: any isometric immersion in a manifold of constant sectional curvature is very special, and each hypersurface of an Einstein manifold is special (see Theorem 1.1 above); any complex, or totally real, immersion in a Kähler manifold with constant holomorphic curvature is very special (see Theorem 1.2 above). Riemannian products of the above examples provide other special (or very special) immersions.

3. PRELIMINARIES

We begin this section proving the following lemma.

Lemma 3.1. *Let $f : S \rightarrow M$ be an isometric immersion where S is an r -dimensional manifold. Then f is special if, and only if, for any $p \in S$ and any local orthonormal frame (X_i) on S which is geodesic at p we have:*

$$(4) \quad r \left(\nabla_{\bar{X}_j}^\perp H \right) (p) = \left(\sum_{i=1}^r \nabla_{\bar{X}_i}^\perp \alpha(X_i, X_j) \right) (p), \text{ for any } j,$$

where r is the dimension of S and $H = \frac{1}{r} \sum_{i=1}^r \alpha(X_i, X_i)$ is the mean curvature vector.

Proof. For local vector fields X, Y, Z on S the Codazzi equation says that

$$(5) \quad (\bar{R}(X, Y)Z)^\perp = (\nabla_X^\perp \alpha)(Y, Z) - (\nabla_Y^\perp \alpha)(X, Z),$$

where

$$(\nabla_X^\perp \alpha)(Y, Z) = \nabla_X^\perp \alpha(Y, Z) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z).$$

Fix $\eta \perp T_p S$ and some local vector field N orthogonal to S satisfying $N(p) = \eta$. Take orthonormal vectors $v_1, \dots, v_r \in T_p S$ and some extension of them to a local orthonormal frame X_1, \dots, X_r on S which is geodesic at p . Fix i, j . At the point p we have:

$$\begin{aligned} \langle \bar{R}(v_i, v_j)v_i, \eta \rangle &= \langle \bar{R}(X_i, X_j)X_i, N \rangle (p) \\ &= \left\langle (\nabla_{X_i}^\perp \alpha)(X_j, X_i) - (\nabla_{X_j}^\perp \alpha)(X_i, X_i), N \right\rangle (p) \\ &= \left\langle \nabla_{X_i}^\perp \alpha(X_j, X_i) - \nabla_{X_j}^\perp \alpha(X_i, X_i), N \right\rangle (p), \end{aligned}$$

where we used the fact that $(\nabla_{X_i} X_j)(p) = (\nabla_{X_i} X_i)(p) = (\nabla_{X_j} X_i)(p) = 0$. By summing up we obtain that

$$(6) \quad \left\langle \sum_i \bar{R}(v_i, v_j)v_i - \left(\sum_i \nabla_{X_i}^\perp \alpha(X_j, X_i) \right) (p) + r \left(\nabla_{\bar{X}_j}^\perp H \right) (p), \eta \right\rangle = 0.$$

From (6) we obtain easily that f is special if, and only if,

$$r \left(\nabla_{X_j}^\perp H \right) (p) = \left(\sum_{i=1}^r \nabla_{X_i}^\perp \alpha(X_i, X_j) \right) (p).$$

Lemma 3.1 is proved. \square

Lemma 3.2. *Equation (1) does not depend on the choice of an orthonormal basis v_1, \dots, v_r of $T_p S$. Similarly (4) does not depend on the choice of a local orthonormal frame X_1, \dots, X_r which is geodesic at p .*

Proof. Fix an orthonormal basis (v_i) of $T_p S$, $w \in T_p S$ and $\eta \in (T_p S)^\perp$. Now consider the linear map $A_{w\eta} : T_p S \rightarrow T_p S$ given by $A_{w\eta}(v) = \pi(\bar{R}(w, v)\eta)$, where $\pi : T_p M \rightarrow T_p S$ is the standard orthogonal projection. For the trace $\text{tr}(A_{w\eta})$ we have:

$$(7) \quad \text{tr}(A_{w\eta}) = \sum_i \langle \bar{R}(w, v_i)\eta, v_i \rangle = \sum_i \langle \bar{R}(v_i, w)v_i, \eta \rangle.$$

From (7) we see that the choice of the orthonormal basis (v_i) is irrelevant on (1). From this and Lemma 3.1 we obtain that the choice of the orthonormal frame (X_i) geodesic at p is irrelevant in Equation (4). \square

By using a proof easier and similar to the proof of Lemma 3.1 we have the following

Lemma 3.3. *Let $f : S \rightarrow M$ be an isometric immersion. We have that f is very special if, and only if, for any $p \in S$ and any local orthonormal fields X, Y on S satisfying $\nabla_X Y(p) = \nabla_Y X(p) = \nabla_X X(p) = \nabla_Y Y(p) = 0$, we have:*

$$(8) \quad (\nabla_Y^\perp \alpha(X, X))(p) = (\nabla_X^\perp \alpha(X, Y))(p).$$

Proposition 3.1. *Let $f : S \rightarrow M$ be a umbilical immersion and r the dimension of S . If $r \geq 2$ then the following conditions are equivalent:*

- (1) f is special;
- (2) f is very special;
- (3) f is an extrinsic sphere.

Proof. Fix $p \in S$ and an orthonormal frame (X_i) in a neighborhood of p which is geodesic at p . Since f is umbilical we have that $\alpha(X, Y) = \langle X, Y \rangle H$. Assume that f is special and fix $j \in \{1, \dots, r\}$. It follows from (4) that

$$r \left(\nabla_{X_j}^\perp H \right) (p) = \left(\sum_{i=1}^r \nabla_{X_i}^\perp \alpha(X_i, X_j) \right) (p) = \left(\sum_{i=1}^r \nabla_{X_i}^\perp (\delta_{ij} H) \right) (p) = \left(\nabla_{X_j}^\perp H \right) (p).$$

Since $r \geq 2$ we have that $\nabla_{X_j}^\perp H = 0$, hence f is an extrinsic sphere.

Now assume that f is an extrinsic sphere. Fix $p \in S$ and local orthonormal fields X, Y on S satisfying $\nabla_X Y(p) = \nabla_Y X(p) = \nabla_X X(p) = \nabla_Y Y(p) = 0$. Then both sides in (8) vanish, hence f is very special. Proposition 3.1 is proved. \square

4. THE RIEMANNIAN CASE - PROOF OF THEOREM 1.1

We first assume that M satisfies the axiom of special r -submanifolds for some $2 \leq r \leq m-2$. To show that the sectional curvature is constant it is sufficient, by Schur's Lemma (see for example [S], II p. 328), to prove that at any point $p \in M$ the sectional curvature is constant for planes contained in $T_p M$. To show this last fact it suffices to obtain that $\langle \bar{R}(v, w)v, \eta \rangle = 0$ for all orthonormal vectors v, w, η (see for example Lemma 1.9 in [D]).

So we fix $p \in M$ and orthonormal vectors $v, w, \eta \in T_p M$. Since $2 \leq r \leq m-2$, we can construct an orthonormal set

$$\{v_1 = v, v_2 = w, v_3, \dots, v_{r+1}, \eta\} \subset T_p M.$$

By our hypothesis there exists a special submanifold S_1 containing p such that

$$T_p(S_1) = \text{span}(v_1, \dots, v_r),$$

where $\text{span}(v_1, \dots, v_r)$ denotes the linear subspace generated by v_1, \dots, v_r . By the definition of a special submanifold we have that

$$(9) \quad \sum_{i=1}^r \langle \bar{R}(v_i, w)v_i, \eta \rangle = 0.$$

Note that $\bar{R}(v_2, w)v_2 = 0$, since $w = v_2$. Fix $1 \leq j \leq r, j \neq 2$. By hypothesis there exists a special submanifold S_2 containing p such that

$$(10) \quad T_p(S_2) = \text{span}(v_1, \dots, \hat{v}_j, \dots, v_r, v_{r+1}),$$

where \hat{v}_j means that the vector v_j is omitted. Since we have that $w = v_2 \in \text{span}(v_1, \dots, \hat{v}_j, \dots, v_r, v_{r+1})$, the definition of a special submanifold implies that

$$(11) \quad \left(\sum_{1 \leq i \leq r, i \neq j} \langle \bar{R}(v_i, w)v_i, \eta \rangle \right) + \langle \bar{R}(v_{r+1}, w)v_{r+1}, \eta \rangle = 0.$$

From (9) and (11) we obtain that $\langle \bar{R}(v_j, w)v_j, \eta \rangle = \langle \bar{R}(v_{r+1}, w)v_{r+1}, \eta \rangle$, for all $1 \leq j \leq r, j \neq 2$. Thus we obtain that

$$(12) \quad 0 = \left\langle \sum_{i=1}^r \bar{R}(v_i, w)v_i, \eta \right\rangle = (r-1) \langle \bar{R}(v_1, w)v_1, \eta \rangle = (r-1) \langle \bar{R}(v, w)v, \eta \rangle,$$

hence $\langle \bar{R}(v, w)v, \eta \rangle = 0$ and M has constant sectional curvature.

Now we assume that M satisfies the axiom of special $(m-1)$ -submanifolds. To show that M is Einstein, it suffices to prove that, for any point $p \in M$ and any orthonormal vectors $w, \eta \in T_p M$, the Ricci bilinear form satisfies

$$(13) \quad -\text{Ric}(w, \eta) = \left\langle \sum_{i=1}^m \bar{R}(v_i, w)v_i, \eta \right\rangle = 0,$$

for some orthonormal basis v_1, \dots, v_m of $T_p M$. Without loss of generality we may assume that $v_m = \eta$, hence $w \in \text{span}(v_1, \dots, v_{m-1})$. By hypothesis there exists a special submanifold S containing p such that $T_p S = \text{span}(v_1, \dots, v_{m-1})$, hence we obtain that $\sum_{i=1}^{m-1} \bar{R}(v_i, w)v_i \in T_p S$, hence (13) holds and M is Einstein.

To finish the proof of Theorem 1.1 we need to show that every isometric immersion in a manifold M of constant sectional curvature is very special, and that every codimension one isometric immersion in an Einstein manifold is special.

First we consider the case that M has constant sectional curvature ρ . Consider an isometric immersion $f : S \rightarrow M$. Fix $p \in S$. Given vectors $v, w \in T_p S$, we have that $\langle \bar{R}(v, w)v, \eta \rangle = \rho(\langle v, \eta \rangle \langle w, v \rangle - \langle v, v \rangle \langle w, \eta \rangle) = 0$ for all $\eta \in (T_p S)^\perp$, hence $\bar{R}(v, w)v \in T_p S$. As a consequence f is very special.

Now we consider the case that M is Einstein. Let $f : S \rightarrow M$ be a codimension one isometric immersion. Fix $p \in S$ and $\eta \in (T_p S)^\perp, |\eta| = 1$. Consider orthonormal vectors $v_1, \dots, v_{m-1} \in T_p S$. Since M is Einstein we have that $\text{Ric}(v, w) = \rho \langle v, w \rangle$ for some $\rho \in \mathbb{R}$. Set $v_m = \eta$. So we have

$$\sum_{i=1}^{m-1} \langle \bar{R}(v_i, w)v_i, \eta \rangle = \sum_{i=1}^m \langle \bar{R}(v_i, w)v_i, \eta \rangle = -\rho \langle w, \eta \rangle = 0,$$

for all $w \in T_p S$. We conclude that f is special. Theorem 1.1 is proved.

5. PROPERTIES OF XY -MANIFOLDS

The following two propositions will be needed in the proof of Theorem 1.2

Proposition 5.1. *Let M be a Kähler manifold of real dimension $2m$, $m \geq 2$, with almost complex structure J . Fix $p \in M$. The following assertions are equivalent:*

(a) *For any orthonormal vectors $X, JX, Y, JY \in T_p M$ it holds that*

$$\langle \bar{R}(X, JX)Y, JX \rangle = \langle \bar{R}(Y, JY)X, JY \rangle;$$

(b) *For any orthonormal vectors $X, JX, Y, JY \in T_p M$ it holds that*

$$\langle \bar{R}(X, JX)Y, X \rangle = -\langle \bar{R}(Y, JY)X, Y \rangle;$$

(c) *For any orthonormal vectors $X, JX, Y, JY \in T_p M$ it holds that*

$$\langle \bar{R}(X, Y)X, JY \rangle = 0.$$

Proof. If we replace Y by JY in (a) we obtain (b). If we replace Y by $-JY$ in (b) we obtain (a).

Now we will prove that (b) and (c) are equivalent. We set $A = \frac{X+Y}{\sqrt{2}}, B = \frac{X-Y}{\sqrt{2}}$, hence $X = \frac{A+B}{\sqrt{2}}, Y = \frac{A-B}{\sqrt{2}}$. Clearly A, JA, B, JB are orthonormal if and only if X, JX, Y, JY are orthonormal. We have:

$$\begin{aligned} 4 \langle \bar{R}(X, Y)X, JY \rangle &= \langle \bar{R}(A+B, A-B)(A+B), JA-JB \rangle \\ &= 2 \langle \bar{R}(B, A)(A+B), JA-JB \rangle \\ &= 2 \{ \langle \bar{R}(B, A)A, JA \rangle - \langle \bar{R}(B, A)B, JB \rangle \} \\ &= 2 \{ \langle \bar{R}(A, JA)B, A \rangle + \langle \bar{R}(B, JB)A, B \rangle \}, \end{aligned}$$

where we used the fact that $\langle \bar{R}(B, A)A, JB \rangle = \langle \bar{R}(B, A)B, JA \rangle$. From the above equation it is clear that (b) is equivalent to (c). \square

Proposition 5.2. *Let M be a XY -manifold of real dimension $2m$ with $m \geq 3$. Fix $p \in T_p M$. Take orthonormal vectors $X, JX, Y, JY, Z, JZ \in T_p M$. Then it holds that*

$$\begin{aligned} \langle \bar{R}(X, JX)Y, JX \rangle &= 2 \langle \bar{R}(Z, JZ)X, JY \rangle \\ &= 4 \langle \bar{R}(Z, X)Z, Y \rangle = 4 \langle \bar{R}(JZ, X)JZ, Y \rangle. \end{aligned}$$

Remark 5. It is very surprising that the right hand side of the above equation does not depend on the choice of the unit vector Z in $\text{span}(X, JX, Y, JY)^\perp$.

Proof of Proposition 5.2. We will apply the definition of XY -manifolds to the orthonormal vectors $X, JX, \frac{Y+Z}{\sqrt{2}}, \frac{JY+JZ}{\sqrt{2}}$, obtaining that

$$\left\langle \bar{R}(X, JX) \frac{Y+Z}{\sqrt{2}}, JX \right\rangle = \left\langle \bar{R} \left(\frac{Y+Z}{\sqrt{2}}, \frac{JY+JZ}{\sqrt{2}} \right) X, \frac{JY+JZ}{\sqrt{2}} \right\rangle,$$

hence

$$\begin{aligned} 2 \langle \bar{R}(X, JX) (Y+Z), JX \rangle &= \langle \bar{R}(Y, JY)X, JY \rangle + \langle \bar{R}(Y, JY)X, JZ \rangle \\ &+ \langle \bar{R}(Y, JZ)X, JY \rangle + \langle \bar{R}(Y, JZ)X, JZ \rangle \\ &+ \langle \bar{R}(Z, JY)X, JY \rangle + \langle \bar{R}(Z, JY)X, JZ \rangle \\ &+ \langle \bar{R}(Z, JZ)X, JY \rangle + \langle \bar{R}(Z, JZ)X, JZ \rangle. \end{aligned}$$

By using again that M is a XY -manifold we may cancel the first and last terms on the right side with corresponding terms on the left side, obtaining:

$$\begin{aligned} \langle \bar{R}(X, JX) (Y+Z), JX \rangle &= \langle \bar{R}(Y, JY)X, JZ \rangle + \langle \bar{R}(Y, JZ)X, JY \rangle \\ &+ \langle \bar{R}(Y, JZ)X, JZ \rangle + \langle \bar{R}(Z, JY)X, JY \rangle \\ &+ \langle \bar{R}(Z, JY)X, JZ \rangle + \langle \bar{R}(Z, JZ)X, JY \rangle. \end{aligned}$$

Now we replace Z by $-Z$ obtaining a new equality, which we add to the above equation and divide by 2. All terms where Z appears just one time will be cancelled. Thus we obtain:

$$(14) \quad \langle \bar{R}(X, JX)Y, JX \rangle = 2 \langle \bar{R}(Y, JZ)X, JZ \rangle + \langle \bar{R}(Z, JZ)X, JY \rangle,$$

where we also used the fact that $\langle \bar{R}(Y, JZ)X, JZ \rangle = \langle \bar{R}(Z, JY)X, JZ \rangle$. Now we change Z by JZ , obtaining that

$$(15) \quad \langle \bar{R}(X, JX)Y, JX \rangle = 2 \langle \bar{R}(Y, Z)X, Z \rangle + \langle \bar{R}(Z, JZ)X, JY \rangle.$$

□

By (14) and (15) we obtain that

$$(16) \quad \langle \bar{R}(Y, JZ)X, JZ \rangle = \langle \bar{R}(Y, Z)X, Z \rangle.$$

By (14) and (16) we have that

$$(17) \quad \langle \bar{R}(X, JX)Y, JX \rangle = \langle \bar{R}(Y, Z)X, Z \rangle + \langle \bar{R}(Y, JZ)X, JZ \rangle + \langle \bar{R}(Z, JZ)X, JY \rangle.$$

By the Bianchi equality it is well known that

$$(18) \quad \langle \bar{R}(Z, JZ)X, JY \rangle = \langle \bar{R}(Y, Z)X, Z \rangle + \langle \bar{R}(Y, JZ)X, JZ \rangle.$$

Thus Proposition 5.2 follows directly from (16), (17) and (18). The following proposition will not be used in the proof of Theorem 1.2.

Proposition 5.3. *Let M be a XY -manifold of real dimension $2m, m \geq 4$. Then it holds that*

$$\langle \bar{R}(X, Y)Z, W \rangle = 0,$$

for any orthonormal vectors $X, JX, Y, JY, Z, JZ, W, JW \in T_p M$ and any $p \in M$.

Proof. Fix orthonormal vectors $X, Y, Z, W \in T_p M$ spanning an antiholomorphic linear subspace of $T_p M$. Consider the function $g_{XY} : D \rightarrow \mathbb{R}$, given by $g_{XY}(U) = \langle \bar{R}(U, JU)X, Y \rangle$, where D is the set of unit vectors in $T_p M$ orthogonal to the vectors X, JX, Y, JY . By Proposition 5.2, the function g_{XY} is constant on its domain, hence we have that

$$0 = (dg_{XY})_Z(JW) = 2 \langle \bar{R}(W, Z)X, Y \rangle,$$

and thus the proof is complete. \square

Remark 6. One could ask if manifolds satisfying the antisymmetry property

$$\langle \bar{R}(X, JX)Y, JX \rangle = -\langle \bar{R}(Y, JY)X, JY \rangle,$$

for any orthonormal tangent vectors X, JX, Y, JY in $T_p M$ and any $p \in M$, could give us another interesting class of Kähler manifolds. However, this class agrees with the manifolds of constant holomorphic sectional curvature when the real dimension is at least 6. Indeed, following the same idea as in [GM] we set $X = \frac{A+B}{\sqrt{2}}$, $Y = \frac{A-B}{\sqrt{2}}$ and apply the above antisymmetry obtaining that $\langle \bar{R}(X, JX)JX, X \rangle = \langle \bar{R}(Y, JY)JY, Y \rangle$ for any orthonormal vectors X, JX, Y, JY , which implies that the holomorphic sectional curvature is constant under this dimension condition.

6. PROOF OF THEOREM 1.2

Consider a Kähler manifold M of real dimension $2m$, $m \geq 2$. If we consider the function $h : S^{m-1} \rightarrow \mathbb{R}$ given by $h(X) = \langle \bar{R}(X, JX)JX, X \rangle$, where S^{m-1} is the unit sphere on $T_p M$, the derivative $dh_X Y = -4 \langle \bar{R}(X, JX)Y, JX \rangle$, hence h is constant if, and only if, $\langle \bar{R}(X, JX)Y, JX \rangle = 0$ for all Y orthogonal to X . Since $T_X(S^{m-1}) = \text{span}(JX) \oplus (\text{span}(X, JX))^\perp$, by using the Kählerian analogue of Schur's Lemma it is easy to obtain the following well known result (see [GM]).

Proposition 6.1. *Let M be a Kähler manifold of real dimension $2m$, $m \geq 2$. Then M has constant holomorphic sectional curvature if, and only if, for any $p \in M$ and any orthonormal vectors $X, JX, Y, JY \in T_p M$ it holds that*

$$(19) \quad \langle \bar{R}(X, JX)Y, JX \rangle = 0.$$

By replacing Y by JY in Proposition 6.1 one obtains the following

Corollary 6.1. *M has constant holomorphic sectional curvature if, and only if, it holds that*

$$(20) \quad \langle \bar{R}(X, JX)Y, X \rangle = 0,$$

for any orthonormal vectors $X, JX, Y, JY \in T_p M$ and any $p \in M$.

Proof of Theorem 1.2. Let M be a Kähler manifold of dimension $2m$, $m \geq 2$. We first assume that M satisfies the axiom of special holomorphic r -submanifolds for some $1 \leq r \leq m-1$. Fix $p \in M$ and orthonormal vectors $X, JX, Y, JY \in T_p M$. By Proposition 6.1 it suffices to show that $\langle \bar{R}(X, JX)Y, JX \rangle = 0$.

If $r = 1$ by hypothesis there exists a special surface S containing p such that $T_p S = \text{span}(X, JX)$. By using the definition of a special surface we have that

$$0 = \langle \bar{R}(X, X)X, Y \rangle + \langle \bar{R}(JX, X)JX, Y \rangle = \langle \bar{R}(X, JX)Y, JX \rangle,$$

hence the holomorphic sectional curvature of M is constant.

Now we assume that $r \geq 2$, hence $m \geq 3$. There exist orthonormal vectors

$$X_1 = X, JX_1, X_2, JX_2, \dots, X_r, JX_r \in (\text{span}(Y, JY))^\perp.$$

Set $W = \text{span}(X_1, JX_1, \dots, X_r, JX_r)$. There exists a special submanifold S_1 containing p such that $T_p(S_1) = W$. Then we have that

$$(21) \quad \langle \bar{R}(JX, X)JX, Y \rangle + \sum_{i=2}^r \left(\langle \bar{R}(X_i, X)X_i, Y \rangle + \langle \bar{R}(JX_i, X)JX_i, Y \rangle \right) = 0.$$

Now set $\Omega = \text{span}(X_2, JX_2, \dots, X_r, JX_r, Y, JY)$. There exists a special submanifold S_2 containing p such that $T_p(S_2) = \Omega$. Then we have that

$$(22) \quad \langle \bar{R}(JY, Y)JY, X \rangle + \sum_{i=2}^r \left(\langle \bar{R}(X_i, Y)X_i, X \rangle + \langle \bar{R}(JX_i, Y)JX_i, X \rangle \right) = 0.$$

From (21) and (22) we see that $\langle \bar{R}(X, JX)Y, JX \rangle = \langle \bar{R}(Y, JY)X, JY \rangle$, hence M is a XY -manifold. By Proposition 5.2 we have that

$$\langle \bar{R}(JX, X)JX, Y \rangle = 2 \left(\langle \bar{R}(X_i, X)X_i, Y \rangle + \langle \bar{R}(JX_i, X)JX_i, Y \rangle \right),$$

for all $2 \leq i \leq r$. This fact together with (21) implies that $\langle \bar{R}(JX, X)JX, Y \rangle = 0$, hence M has constant holomorphic sectional curvature.

Now we will assume that M satisfies the axiom of special antiholomorphic r -submanifolds for some $2 \leq r \leq m$. Again we fix $p \in M$ and orthonormal vectors $X, JX, Y, JY \in T_p M$. We consider orthonormal vectors $X_1 = X, X_2 = Y, \dots, X_r$ spanning an antiholomorphic subspace W of $T_p M$. There exists a special submanifold S_1 containing p satisfying $T_p(S_1) = W$. Thus we have that:

$$(23) \quad \langle \bar{R}(X, Y)X, JY \rangle + \sum_{i=2}^r \langle \bar{R}(X_i, Y)X_i, JY \rangle = 0,$$

and

$$(24) \quad \langle \bar{R}(X, Y)X, JX \rangle + \sum_{i=2}^r \langle \bar{R}(X_i, Y)X_i, JX \rangle = 0.$$

Now set $\Omega = \text{span}(JX, X_2, \dots, X_r)$. Let S_2 be a special submanifold satisfying $T_p(S_2) = \Omega$. We have:

$$(25) \quad \langle \bar{R}(JX, Y)JX, JY \rangle + \sum_{i=2}^r \langle \bar{R}(X_i, Y)X_i, JY \rangle = 0.$$

From (23) and (25) it follows that

$$\langle \bar{R}(X, Y)X, JY \rangle = \langle \bar{R}(JX, Y)JX, JY \rangle = -\langle \bar{R}(X, Y)X, JY \rangle,$$

hence $\langle \bar{R}(X, Y)X, JY \rangle = 0$. Since X, Y are arbitrary we conclude by Proposition 5.1 that M is a XY -manifold.

If $m = 2$ then $r = 2$ and it follows immediately from (24) that $\langle \bar{R}(X, Y)X, JX \rangle = 0$, hence Corollary 6.1 implies that M has constant holomorphic sectional curvature. Now assume that $m \geq 3$. We apply Propositions 5.2 and 5.1 to each term

$\langle \bar{R}(X_i, Y)X_i, JX \rangle$ in (24), with $3 \leq i \leq m$, obtaining that

$$\begin{aligned} 4 \langle \bar{R}(X_i, Y)X_i, JX \rangle &= \langle \bar{R}(Y, JY)JX, JY \rangle \\ &= \langle \bar{R}(Y, JY)X, Y \rangle \\ &= -\langle \bar{R}(X, JX)Y, X \rangle \\ &= \langle \bar{R}(X, Y)X, JX \rangle. \end{aligned}$$

By this equality we obtain from (24) that $\langle \bar{R}(X, Y)X, JX \rangle = 0$, hence we conclude again that M has constant holomorphic sectional curvature.

To conclude the proof of Theorem 1.2 we need to prove that complex and totally real immersions in a Kähler manifold of constant holomorphic sectional curvature are very special. In fact, if M has constant holomorphic sectional curvature and X, JX, Y, JY are orthonormal vectors in $T_p M$ for some point $p \in M$, we use together Propositions 5.1, 6.1 and Corollary 6.1, obtaining that

$$(26) \quad 0 = \langle \bar{R}(X, JX)Y, JX \rangle = \langle \bar{R}(X, JX)Y, X \rangle = \langle \bar{R}(X, Y)X, JY \rangle.$$

If furthermore $Z \in (\text{span}(X, JX, Y, JY))^\perp$ we use Propositions 5.2 and 6.1, as well as Corollary 6.1 obtaining that

$$(27) \quad 0 = \langle \bar{R}(Z, X)Z, Y \rangle = \langle \bar{R}(JZ, X)JZ, Y \rangle = \langle \bar{R}(Z, JX)Z, Y \rangle.$$

Let $f : S \rightarrow M$ be a complex immersion in a Kähler manifold of constant holomorphic sectional curvature. Take a point $p \in S$ and fix $Z \in (T_p S)^\perp$ and $X \in T_p S$. We have by (26) that $\langle \bar{R}(X, JX)X, Z \rangle = \langle \bar{R}(JX, X)JX, Z \rangle = 0$. Thus by linearity we see that S is very special if its real dimension is 2. If the real dimension of S is at least 4, we fix $Y \in T_p S$ orthogonal to X, JX . Since f is a complex immersion we have that $Z \in (\text{span}(X, JX, Y, JY))^\perp$, hence we obtain from (27) that

$$\langle \bar{R}(X, Y)X, Z \rangle = \langle \bar{R}(JX, Y)JX, Z \rangle = \langle \bar{R}(Y, X)Y, Z \rangle = \langle \bar{R}(Y, JX)Y, Z \rangle = 0.$$

By the linearity properties of the curvature tensor in a Kähler manifold we conclude that f is very special.

Now let $f : S \rightarrow M$ be a totally real immersion, where M has constant holomorphic sectional curvature. Take $p \in S$ and orthonormal vectors $X, Y \in T_p S$. By (26) we have that $\langle \bar{R}(X, Y)X, JX \rangle = 0$ and $\langle \bar{R}(X, Y)X, JY \rangle = 0$. If $Z \in (T_p S)^\perp$ is orthogonal to X, JX, Y, JY we obtain from (27) that $\langle \bar{R}(X, Y)X, Z \rangle = 0$. By interchanging X and Y we obtain similar equations. Thus we may use the linearity properties of the curvature tensor in a Kähler manifold to conclude that f is very special.

Theorem 1.2 is proved. \square

REFERENCES

- [Ca] Cartan, E., *Leçons sur la Géométrie des Espaces de Riemann*, Gauthier-Villars, Paris, (1951).
- [CO] Chen, B.-Y., Ogiue, K., *Some characterizations of complex space forms*, Duke Math. J., **40** (1973), 797–799.
- [D] Dajczer, *Submanifolds and isometric immersions*, Mathematics Lecture Series, Series Editor Richard S. Palais.
- [G] Goldberg, S. I., *The axiom of 2-spheres in Kaehler geometry*, J. of Differential Geometry, **8** (1973), 177–179.
- [GM] Goldberg, S. I., Moskal, E. M., *The axiom of spheres in Kaehler geometry*, Kodai Math. Sem. Rep., **27** (1976), 188–192.

- [H] Harada, M. *On Kaehler manifolds satisfying the axiom of antiholomorphic 2-spheres*, Proc. of A.M.S., **43** (1974), 186–189.
- [MY] Mogi, I., Yano, K., *On real representations of Kaehlerian manifolds*, Annals of Mathematics, **61** No. 1 (1955), 170–189.
- [LN] Leung, D., Nomizu, K., *The axiom of spheres in Riemannian Geometry*, J. Diff. Geometry, **5**, (1971), 487–489.
- [S] Spivak, M., *A comprehensive introduction to differential geometry*. Publish or Perish Inc., Houston, 1979.

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